A strain-displacement variational formulation for laminated composite beams based on the modified couple stress theory

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1. Introduction

Mixed energy methods, *e.g.* Hellinger-Reissner (HR) method, can include stresses and displacements as unknowns in the formulation and can be made to satisfy equilibrium conditions as well as minimise total energy. The HR method accurately captures the full 3D stress field of general beams and plates [1].

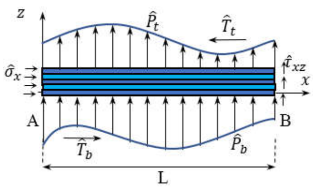
The differential quadrature method (DQM) can be viewed as a more accurate, computationally efficient version of the finite difference method. It solves differential equations directly and therefore solves pointwise equilibrium expressions and can be more accurate than conventional FE methods.

The present study aims to:

- develop strain-displacement (SD) mixed energy method for stress analysis of laminated beams.
- include couple-stress in the SD formulation to accurately predict 3D stresses of highly heterogeneous laminates.
- implement DQM method to solve the governing equations subject to different boundary conditions.

2. Mathematical formulation

Consider a laminated beam with the length and the rectangular cross-section being L and $b \times t$. The beam is constituted of N layers and fibre angle (in stacking sequence) $\theta^{(k)}$ in kth layer is referred to x-axis. Equivalent single layer (ESL) theory is used.



Beam's configuration

Displacement profile (Third-order zigzag): $u_x^{(k)}(x,z) = \mathbf{f}_{\phi_u}^{(k)} \boldsymbol{U}$, $u_z(x) = w_0$

Micro-rotation: $\theta_y^{(k)} = \frac{1}{2}\left(\mathbf{f}_{\phi_u,z}^{(k)} - w_{0,x}\right)$

Axial strain: $\varepsilon_x^{(k)} = \mathbf{f}_{\phi_u}^{(k)} \boldsymbol{U}_{,x} = \mathbf{f}_{\phi_u}^{(k)} \boldsymbol{\varepsilon}$

Micro-curvature: $\chi_{xy}^{(k)} = \frac{1}{2}\mathbf{f}_{c_1}^{(k)} \boldsymbol{\chi}_{xy}$

Axial stress: $\sigma_x^{(k)} = \bar{Q}_{11}^{(k)} \mathbf{f}_{\phi_u}^{(k)} \boldsymbol{\varepsilon}$

Couple stress: $m_{xy}^{(k)} = \frac{1}{2}\tilde{Q}_{44}^{(k)} \mathbf{f}_{c_1}^{(k)} \boldsymbol{\chi}_{xy}$

Stress resultant (by strain vector): $\mathcal{F} = \int_{-t/2}^{t/2} \mathbf{f}_{\phi_u}^{(k)^T} \bar{Q}_{11}^{(k)} \mathbf{f}_{\phi_u}^{(k)} \, dz = \mathbf{S}\boldsymbol{\varepsilon}$

Couple stress resultant (by micro-curvature): $\mathcal{M}_{xy} = \int_{-t/2}^{t/2} \mathbf{f}_{c_1}^{(k)^T} \tilde{Q}_{44}^{(k)} \mathbf{f}_{c_1}^{(k)^T} \, dz = \boldsymbol{S}_{c_1} \boldsymbol{\chi}_{xy}$

Transverse shear & normal stresses — Equilibrium in Modified couple stress theory (MCST)

Integration through thickness (ESL) — Force equilibrium

System total energy — Lagrange multipliers (corresponding to displacements)

Variational principle is performed with respect to Strain and Displacement functionals for Governing and Boundary equations (SD MCS model).
A reduced strain-displacement (SD) model without couple stress is obtained by removing the micro-curvature vector.

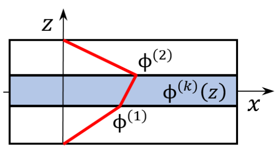
Third-order zigzag displacement profile [1] :

$$u_x^{(k)}(x,z) = u_0 + z\theta + z^2\zeta + z^3\xi + \phi^{(k)}(z)\psi$$
$$u_z(x) = w_0$$

Refined zigzag function $\phi^{(k)}(z)$ [2]:

$$\phi^{(1)}(z) = \left(z + \frac{t}{2}\right)\left(\frac{G}{G_{xz}^{(1)}} - 1\right)$$

$$\phi^{(k)}(z) = \left(z + \frac{t}{2}\right)\left(\frac{G}{G_{xz}^{(k)}} - 1\right) + \sum_{i=2}^{k} t^{(i-1)}\left(\frac{G}{G_{xz}^{(i-1)}} - \frac{G}{G_{xz}^{(k)}}\right)$$



Zigzag profile

where $G_{xz}^{(k)}$ is the in-plane shear modulus of layer k, G the equivalent shear modulus of laminate.

Equilibrium conditions in the MCST [3]:

$$\sigma_{x,x}^{(k)} + \tau_{xz,z}^{(k)} - \frac{1}{2}\left(m_{xy,xz}^{(k)} + m_{zy,zz}^{(k)}\right) = 0$$

$$\tau_{xz,x}^{(k)} + \sigma_{z,z}^{(k)} + \frac{1}{2}\left(m_{xy,xx}^{(k)} + m_{zy,xz}^{(k)}\right) = 0$$

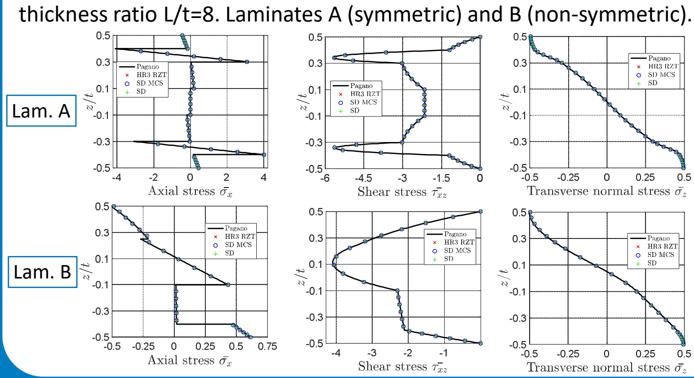
3. Results

Material properties and Laminate configuration:

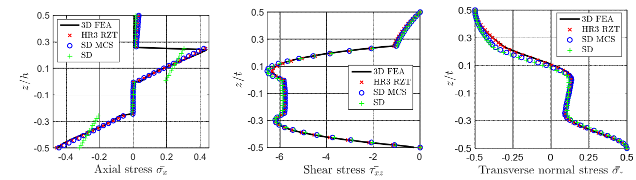
Mat.	$\frac{E_1}{G_{12}^{(h)}}$	$\frac{E_2}{G_{12}^{(h)}}$	$\frac{E_3}{G_{12}^{(h)}}$	$\frac{G_{12}}{G_{12}^{(h)}}$	$\frac{G_{13}}{G_{12}^{(h)}}$	$\frac{G_{23}}{G_{12}^{(h)}}$
h	250	250	2500	1	875	1750
p	25×10^6	1×10^6	1×10^6	5×10^5	5×10^5	2×10^5
m	32.57×10^6	1×10^6	10×10^6	6.5×10^5	8.21×10^6	3.28×10^6
pvc	25×10^4	25×10^4	25×10^4	9.62×10^4	9.62×10^4	9.62×10^4

Lam.	Layer thickness ratio	Layer materials	Stacking sequence
A	$[0.1_2/0.2_3/0.1_2]$	$[p_2/pvc/h/pvc/p_2]$	$[90/0_5/90]$
B	[0.1/0.3/0.35/0.25]	$[p_2/m/p]$	$[0/90/0_2]$
1	$[0.25_4]$	$[p_4]$	[0/90/0/90]
2	$[(1/8)_2/0.5/(1/8)_2]$	$[p_2/pvc/p_2]$	$[0/90/0_2/90]$

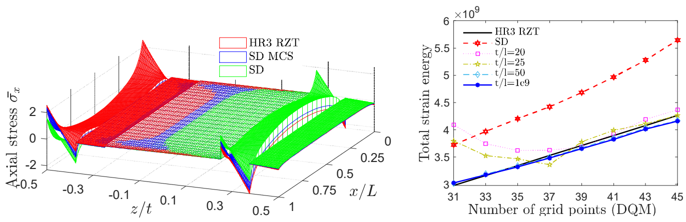
Example 1: Simply supported beam under sinusoidal load, the span-to-thickness ratio L/t=8. Laminates A (symmetric) and B (non-symmetric).



Example 2: Clamped beam under uniformly distributed load, the span-to-thickness ratio L/t=10. Laminates 1 (non-symmetric) and Laminate 2 (sandwich).



Laminate 1: stresses are plotted at position 10% from the left clamp.



Laminate 2: through-thickness axial stress along the beam.

4. Conclusions

- Both the SD and SD MCS models can accurately capture the flexural behaviour of various simply-supported beams including symmetric and antisymmetric laminates, as well as thick-soft core sandwich beams.
- The SD MCS model including the couple stress can predict localised stress efficiently near the clamped boundary, which is challenging for ESL models. (31x8 = 248 DOFs in SD MCS vs. 96,000 C3D8R brick elements in 3D FEA.)
- Changing the length scale parameter, *i.e.* $\tilde{Q}_{44}^{(k)}$, in couple stress can result in better optimisation of system total energy.

References

[1] Groh RMJ, Weaver PM. International Journal of Solids and Structures. 2015;59:147-70.

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[3] Kwon Y-R, Lee B-C. Computational Mechanics. 2016;59(1):117-28.